University of California, Berkeley
Physics H7C Fall 1999 (*Strovink*)

EXAMINATION 1

Directions. Do both problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he or she will not give hints and will be obliged to write your question and its answer on the board. Don't use a calculator, which you don't need – roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (58 points)
In a free-electron laser, a beam of relativistic electrons is subjected to a transverse magnetic field that varies sinusoidally with lab coordinate z, the (average) beam direction:

$$\mathbf{B} = \hat{\mathbf{x}} B_0 \cos \frac{2\pi z}{\lambda_0}$$

where B_0 and λ_0 are constants. In the lab, the z component of the electrons' velocity is

$$v_z = \beta_0 c$$

where β_0 is a constant.

a. (8 points) Consider a Lorentz frame $\mathcal{S}'$ moving with velocity

$$\boldsymbol{\beta}_0 c = \hat{\mathbf{z}} \beta_0 c$$

with respect to the lab. The Lorentz transformation for electromagnetic fields is

$$\begin{aligned}
\mathbf{E}'_\parallel &= \mathbf{E}_\parallel \\
\mathbf{B}'_\parallel &= \mathbf{B}_\parallel \\
\mathbf{E}'_\perp &= \gamma_0(\mathbf{E}_\perp + \boldsymbol{\beta}_0 \times c\mathbf{B}_\perp) \\
c\mathbf{B}'_\perp &= \gamma_0(c\mathbf{B}_\perp - \boldsymbol{\beta}_0 \times \mathbf{E}_\perp) \ ,
\end{aligned}$$

where $\gamma_0 \equiv (1-\beta_0^2)^{-1/2}$. Calculate the electric field $\mathbf{E}'$ seen in $\mathcal{S}'$; continue to express it in terms of $2\pi z/\lambda_0$.

b. (8 points) Defining

$$2\pi z/\lambda_0 \equiv \omega'_0 t' \ ,$$

where t' is the time as observed at the origin of $\mathcal{S}'$, compute ω'_0 in terms of the constants previously given.

c. (8 points) Consider an electron of charge $-e$ and mass m whose *average* position is

$$\langle x', y', z' \rangle = (0, 0, 0)$$

as observed in $\mathcal{S}'$. In this frame, its velocity is so small that you may ignore $\mathbf{v}' \times \mathbf{B}'$ with respect to $\mathbf{E}'$. In frame $\mathcal{S}'$, making this approximation, compute the electron's motion $y'(t')$. (In case you didn't get part **b.** exactly right, leave your answer in terms of ω'_0.)

d. (8 points) The electric dipole moment $\mathbf{p}$ of a distribution of N point charges q_i at positions $\mathbf{r_i}$ is defined as

$$\mathbf{p} = \sum_{i=1}^{N} \mathbf{r}_i q_i \ .$$

The power $P(t)$ radiated by a charge distribution with time-varying dipole moment $\mathbf{p}(t)$ is

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3} \frac{(d^2\mathbf{p}/dt^2)^2}{c^3} \ .$$

As seen in $\mathcal{S}'$, calculate $\langle P' \rangle$, the *time-averaged* power radiated by a single electron in the free-electron laser.

e. (8 points) Energy and time both transform as the 0^{th} component of a four-vector. Calculate $\langle P \rangle$, the time-averaged power radiated by a single electron as observed in the lab. You may leave your answer in terms of $\langle P' \rangle$.

f. (10 points) For light, the relativistic Doppler shift is

$$\omega = \frac{\omega'}{\gamma_0(1 - \beta_0 \cos\theta)} \ .$$

Calculate the ratio λ_0/λ, where λ_0, as before, is the characteristic length describing the spatial variation in the lab of the free-electron laser's magnetic field, and λ is the wavelength of the light that its electrons radiate in the forward direction, as observed in the lab. Express this ratio in terms of γ_0, in the limit $\beta_0 \to 1$.

g. (8 points) What is the state of polarization of the free-electron laser's light? Explain.

2. (42 points)
Semi-infinite regions $y > L$ and $y < -L$ are filled by perfect conductor, while the intervening slab $-L < y < L$ is filled by dielectric with constant dielectric constant ϵ and permeability μ.

a. (8 points) In SI units, write Maxwell's equations for $\mathbf{E}$ and $\mathbf{H}$ inside the dielectric. Do not write any terms involving free charges or free currents, which both vanish there.

b. (8 points) Prove that E_x and E_z both must vanish at $y = \pm L$.

For parts **c.** and **d.** only, assume, for $-L < y < L$, that the fields are given by

$$\begin{aligned}\mathbf{E}_{\text{physical}} &= \Re\big(\hat{\mathbf{y}} E_2 \exp\big(i(kz - \omega t)\big)\big)\\ \mathbf{H}_{\text{physical}} &= \Re\big((\hat{\mathbf{x}} H_1 + \hat{\mathbf{z}} H_3) \exp\big(i(kz - \omega t)\big)\big) \ ,\end{aligned}$$

where E_2, H_1, and H_3 are unknown complex constants, and k and ω are unknown real constants.

c. (8 points) Prove that $H_3 = 0$.

d. (10 points) Calculate the ratio H_1/E_2 in terms of known quantities.

e. (8 points) Can a linear combination of a right-hand and a left-hand circularly polarized plane wave in the region $-L < y < L$ propagate in the z direction? If not, why not? If so, what combination(s) would be possible? Explain fully.